

Solved by
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MTH 320

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The award is for STUDENTS only. If others like to try it just for fun, I will be happy to look at your solution

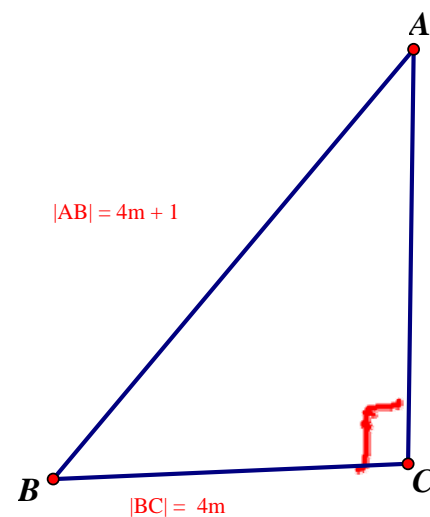
99.99 AED AWARD. Let m be a positive integer. We want to construct a right triangle ACB , where $|AB| = 4m + 1$, $|CB| = 4m$, and $|AC| =$ some positive integer (i.e., all three sides are positive integers). Let S be the set of all possible values of m . Prove that S is a union of k disjoint sets, say S_1, S_2, \dots, S_k , where $S_1 = \{a_1x^2 + b_1x + c_1 \mid x \in \mathbb{N}^*\}$ for some fixed positive integers a_1, b_1, c_1 , and for each $2 \leq i \leq k$ we have $S_i = \{a_ix^2 + b_ix + c_i \mid x \in \mathbb{N}\}$ for some fixed positive integers a_i, b_i, c_i .

Remark: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$

So you need to tell me the exact value of k , and for each $1 \leq i \leq k$ you need to tell me the exact values of a_i, b_i, c_i .

Students in Discrete Math. or Abstract Algebra should know (I guess) how to attack this question. Only very basic elementary number theory is needed here.

As usual: Calculators, Try and Error, and Computer programs are NOT ACCEPTED. You need to give me a correct mathematical argument that clarify your solution



let $m \in \mathbb{N}^*$, $|AB| = 4m+1$, $|CB| = 4m$

$\Rightarrow |AC|^2 = (4m+1)^2 - (4m)^2 = 8m+1$

since $m \in \mathbb{N}^*$ and $|AC|^2 = 8m+1$, $|AC|^2$ is odd $\Rightarrow |AC|$ is odd.
 Any odd number can be uniquely expressed either as $4x+1$ or $4x+3$ for some integer x . Since we are interested in finding all possible values for m , we will consider both cases:

Case 1: $|AC| = 4x+1$ where $\boxed{x > 0}$ since $|AC| \geq 1$ (otherwise m would be positive)

$\Rightarrow |AC|^2 = 16x^2 + 8x + 1 = 8m+1 \Rightarrow m = 2x^2 + x$

Case 2: $|AC| = 4x+3$ where $\boxed{x \geq 0}$

$\Rightarrow |AC|^2 = 16x^2 + 24x + 9 = 8m+1 \Rightarrow m = 2x^2 + 3x + 1$

Note: there is no odd number that can take the both representations $(4x+1)$, $(4x+3)$ together which make the generated values in case 1 and case 2 form two disjoint sets, and since $\frac{|AC|^2 - 1}{8} = m$ and $|AC|$ is positive different $|AC|$ correspond to different m . Thus,

$S_1 = \{ 2x^2 + x \mid x \in \mathbb{N}^* \}$, $S_2 = \{ 2x^2 + 3x + 1 \mid x \in \mathbb{N} \}$

are disjoint and since all possible cases are considered

$S = S_1 \cup S_2$ consist of all the possible values for m .

Thus S is a union of k disjoint sets with the structure given in the question. ▢